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Distributions on Surfaces

by

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§ 0. This topic has not yet, so far as we know, received a very thorough treatment in the literature, but it shows promise of becoming a useful tool. The classical solution of the wave equation by Hadamard (for uneven space dimensions) can be given in such terms; and they are used (if somewhat loosely) to describe the solutions of the Klein-Gordon equation $\sum \partial^2 u/\partial x_i^2 - \partial^2 u/\partial t^2 - m^2 u = 0$ desired in quantum field theory.

This report contains a cleaned-up version of pp. 200-227 of Verallgemeinerte Funktionen I, by Gelfand and Shilov, with a few additions.

The principle difference is in the definition of the distributions in question, $\delta^{(k)}(P)$. The present definition leads to much simpler proofs of existence and uniqueness of the $\delta^{(k)}(P)$, as well as of the relevant formulas. The only differential geometry required is Gauss' theorem in R_n.

§ 1. We consider distributions concentrated on a surface in R_n of dimension n-1 given by an equation $P(x)=0$, where $x=(x_1,\dots,x_n)$, P is a C^∞ function, and ∇P never vanishes on $\{P=0\}$.

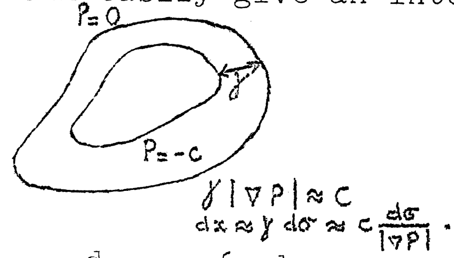
The simplest case is n=1, $P(x)=x$; the distributions concentrated on $\{x=0\}$ are linear combinations of $\delta(x)$, $\delta'(x)$, etc. These are all derivatives of the regular distribution $\theta(x)$: $(\theta(x),\varphi)=\int_{x\geq 0}\varphi(x)dx$. If we now let P be a more general C^∞ function with no double zeros, then $\theta(P)$ is easy to define: $(\theta(P),\varphi)=\int_{P(x)\geq 0}\varphi(x)dx$. $\delta(P)$ should in some sense be the derivative of θ , in fact it is $d\theta(P)/dP = d\,\theta(P)/dx \cdot {}^{dx}/dP$. Thus $(\delta(P),\varphi)=\sum_{x=z_n}\varphi(x)/|P'(x)|$, where z_n are the zeros of P. Similarly $\delta^{(k)}(P)=d^{k+1}\,\theta(P)/dP^{k+1}$.

§ 2. Leaving the case n=1, we find that $\theta(P)$ is still easy to define by $(\theta(P),\varphi)=\int_{P\geq 0}\varphi$. $\delta(P)$ should be in some sense the derivative of $\theta(P)$ with respect to P. The most naive interpretation of this turns out to serve very well.

Definition. If P is a C^{∞} function with ∇P never zero on $\{P=0\}$, then $\delta'(P)=\lim\limits_{c\to 0}\frac{1}{c}\left[\delta(P+c)-\delta(P)\right]$, and

$\delta^{(k+1)}(P)=\lim\limits_{c\to 0}\frac{1}{c}\left[\delta^{(k)}(P+c)-\delta^{(k)}(P)\right]$.

It is not immediately clear that any of these distributions exist, but we can easily give an interpretation of $\delta(P)$. Let $d\sigma$ be the surface measure on $\{P=0\}$.



Then we have $(\delta(P),\varphi)=$

$$\lim_{c\to 0+}\frac{1}{c}\int_{-c\leq P\leq 0}\varphi\,dx \approx \lim\frac{1}{c}\int_{P=0}\varphi\, c\,\frac{d\sigma}{|\nabla P|}=\int_{P=0}\varphi\frac{d\sigma}{|\nabla P|}.$$

This agrees with the case $n=1$ considered above.

§ 3. We establish the existence of $\delta^{(k)}(P)$ by giving it a concrete representation in terms of suitably chosen local coordinates. In some neighbourhood V of any point on $\{P=0\}$ we can introduce new coordinates $u=u(x)$ such that $u_1=P$, the Jacobean $\frac{\partial u}{\partial x}\neq 0$, and the x can also be written uniquely as a function $x(u)$ of u. If $\partial P/\partial x_1$ is not zero in V, for instance, we can choose $u_1=P$, $u_2=x_2,\ldots,u_n=x_n$; and since one of the $\partial P/\partial x_j$ is non-zero in a sufficiently small neighbourhood of any point on $\{P=0\}$, some such system is always available. For a test function φ with support in V we have

$$(\theta(P),\varphi)=\int_{P\geq 0}\varphi\,dx=\int_{u_1\geq 0}\varphi\left|\frac{\partial x}{\partial u}\right|du, \text{ and}$$

$$\lim_{c\to 0}\frac{1}{c}(\theta(P+c)-\theta(P),\varphi)=\lim_{c\to 0}\frac{1}{c}\int_{-c_1\leq u_1\leq 0}\varphi\left|\frac{\partial x}{\partial u}\right|du_1\ldots du_n=$$

$$=\int\left[\varphi\left|\frac{\partial x}{\partial u}\right|\right]_{u_1=0}du_2\ldots du_n. \text{ Further } (\delta^{(1)}(P),\varphi)=$$

$$\lim_{c\to 0}\int\frac{\left[\varphi\left|\frac{\partial x}{\partial u}\right|\right]_{u_1=-c}-\left[\varphi\left|\frac{\partial x}{\partial u}\right|\right]_{u_1=0}}{c}\,du_2\ldots du_n$$

$$= - \int_{u_1=0} \left[\partial(\varphi|\tfrac{\partial x}{\partial u}|)/\partial u_1 \right]_{u_1=0} du_2 \ldots du_n, \text{ and}$$

generally

$$(\delta^{(k)}(P),\varphi)=(-1)^k\int \left[\partial^{(k)}(\varphi|\tfrac{\partial x}{\partial u}|)/\partial u_1^k \right]_{u_1=0} du_2 \ldots du_n.$$

Thus $\delta(P)$ is a simple layer, and $\delta^{(k)}(P)$ a (k+1)-fold layer, roughly speaking. The expression for $\delta(P)$ gives a strict justification of the formula obtained in (§ 2).

§ 4. Examples

i) $P(x_1,\ldots,x_n)=x_1$. $\delta^{(k)}(x_1)=\int \left[\partial^k\varphi/\partial x_1{}^k\right]_{x_1=0} dx_2 \ldots dx_n$.

ii) $P(x_1,\ldots,x_n)=r-c$, $r^2=\sum_1^n x_j{}^2$. Since $|\nabla P|=1$, we have $(\delta(r-c),\varphi)=\int_{r=c}\varphi\, d\sigma$, with $d\sigma$ the surface area on $r = c$, i.e. $d\sigma = c^{n-1}d\Omega$, where $d\Omega$ is the surface area on the unit sphere Ω. Let $x=r\omega$ with $|\omega|=1$. Then finally $(\delta(P),\varphi)=\int_\Omega \varphi(c\omega)c^{n-1}\, d\Omega$. Using the definition in (§ 2),

$$(\delta'(P),\varphi)= \lim_{\varepsilon\to 0}\int_\Omega \frac{1}{\varepsilon}\left[\varphi((c-\varepsilon)\omega)(c-\varepsilon)^{n-1}-\varphi(c\omega)c^{n-1}\right] d\Omega$$

$$= -\int_\Omega \left[\partial\varphi(r\omega)r^{n-1}/\partial r\right]_{r=c} d\Omega,$$

and generally

$$(\delta^{(k)}(P),\varphi)=(-1)^k\int_\Omega \left[\partial^k\varphi(r\omega)r^{n-1}/\partial r^k\right]_{r=c} d\Omega.$$

iii) $P=r^2-c^2$. Proceding as in (ii),

$$(\delta(P),\varphi)=\tfrac{1}{2}\int_\Omega \varphi(c\omega)c^{n-2}d\Omega,$$

$$(\delta'(P),\varphi)=\lim \tfrac{1}{2}\int_\Omega \frac{1}{\varepsilon}\left[\varphi(\omega\sqrt{c^2-\varepsilon})(c^2-\varepsilon)^{n-2}-\varphi(c\omega)c^{n-2}\right] d\Omega$$

$$= \tfrac{1}{2}\int_\Omega \left[\frac{1}{2r}\partial\varphi(r\omega)r^{n-2}/\partial r\right]_{r=c} d\Omega, \text{ and generally}$$

$$(\delta^{(k)}(P),\varphi)= \tfrac{1}{2}\int_\Omega \left[\left(\frac{1}{2r}\frac{\partial}{\partial r}\right)^k \varphi(r\omega)r^{n-2}\right]_{r=c} d\Omega.$$

iv) $P=x_1^2-\sum_2^n x_j^2-c^2$. $\{P=0\}$ is a hyperboloid, on each sheet of which we can set $u_1=P$, $u_2=x_2,\dots,u_n=x_n$, and apply the formula of (§ 3). We have for the Jacobean $\partial u/\partial x=2x_1$, and $\frac{\partial}{\partial u_1}=\sum\frac{\partial x_j}{\partial u_1}\frac{\partial}{\partial x_j}=\frac{1}{2x_1}\frac{\partial}{\partial x_1}$. Thus

$$(\delta^{(k)}(P),\varphi)=$$

$$(-1)^k\int_{-\infty}^{\infty}\!\!\int\Big[\Big(\frac{\partial}{2x_1\partial x_1}\Big)^k(\varphi(x_1,\dots,x_n)/2|x_1|)\Big]_{x_1=\sqrt{\sum_2^n x_j^2+c^2}}$$

$$+\Big[\Big(\frac{\partial}{2x_1\partial x_1}\Big)^k(\varphi(x_1,\dots,x_n)/2|x_1|)\Big]_{x_1=-\sqrt{\sum_2^n x_j^2+c^2}}dx_2..dx_n$$

$$=(-1)^k\int_{-\infty}^{\infty}\!\!\cdots\!\int\Big(\frac{1}{2x_1}\frac{\partial}{\partial x_1}\Big)^k(\varphi(x)/2x_1)\Big|_{x_1=-\sqrt{\;}}^{x_1=\sqrt{\;}}dx_2\dots dx_n.$$

This could also be written as

$$(\delta^{(k)}(P),\varphi)=(-1)^k\int_{|x_1|<\sqrt{\sum_2^n x_j^2+c^2}}\Big(\frac{\partial}{\partial x_1}\frac{1}{2x_1}\Big)^{k+1}\varphi(x)dx_2\dots dx_n$$

$$=-\Big(\theta(-P),\Big(-\frac{\partial}{\partial x_1}\frac{1}{2x_1}\Big)^{k+1}\varphi\Big),$$ which gives a

regularization of $\delta^{(k)}(x_1^2-\sum_2^n x_j^2-c^2)$.

§ 5. Chain rules for $\delta^{(k)}(P)$.

The significance of $\delta^{(k)}(P)$ lies primarily in the chain rule $\partial\delta^{(k)}(P)/\partial x_j=\delta^{(k+1)}(P)\,\partial P/\partial x_j$. Except for this, it would have little advantage over the standard "layers" defined on a surface S independently of its representation, i.e. $(L^{(k)}(S),\varphi)=(-1)^k\int_S\partial^k\varphi/\partial\nu^k\,d\sigma$, with $\partial/\partial\nu$ the normal derivative.

We establish the chain rule inductively, starting with $\theta(P)=\delta^{(-1)}(P)$. Let $\vec{\varphi}_j=(0,\dots,\varphi,\dots,0)$, with the non-zero entry in place j. Then

$$(\partial\theta(P)/\partial x_j,\varphi)=-\int_{P\geq 0}\partial\varphi/\partial x_j dx=-\int_{P\geq 0}\nabla\cdot\vec{\varphi}_j dx=-\int_{P=0}\vec{\varphi}_j\cdot\vec{n}\,d\sigma,$$

with $\vec{n}=-\nabla P/|\nabla P|$ the outer normal to the boundary of

$P \geqslant 0$. Thus $(\partial \theta(P)/\partial x_j, \varphi) = \int_{P=0} P_j \varphi \frac{d\sigma}{|\nabla P|} = (P_j\, \delta(P), \varphi)$, as desired. In general, by induction,

$$(\partial \theta^{(k)}(P)/\partial x_j, \varphi) = -\lim_{c \to 0} \frac{1}{c} (\delta^{(k-1)}(P+c) - \delta^{(k-1)}(P), \partial\varphi/\partial x_j)$$

$$= \lim_{c \to 0} \frac{1}{c} (\delta^{(k)}(P+c) - \delta^{(k)}(P), P_j\, \varphi) = (P_j \delta^{(k+1)}(P), \varphi).$$

§ 6. Another formula

We have $(P\delta(P), \varphi) = \int_{P=0} \frac{P\varphi}{|\nabla P|}\, d\sigma = 0$. Differentiating,

$P_j \delta(P) + P_j P \delta'(P) = 0$, so $|\nabla P|^2 \left[\delta(P) + P\delta'(P)\right] = 0$, and $\delta(P) + P\delta'(P) = 0$. By further differentiating we find

$$\underline{P\delta^{(k)}(P) = -k\,\delta^{(k-1)}(P).}$$

§ 7. Before we can apply the $\delta^{(k)}$ to solving the wave equation, we need a rule for differentiating $\delta(P)$ with respect to a parameter.

Consider a function $P(x,t) = P_t(x)$. Suppose that for $a < t < b$, $P(x,t)$ is C^{∞} in its n+1 variables, and $|P|^2 + \sum_1^n |\partial P/\partial x_j|^2 > 0$. Then we can introduce local coordinates in n+1 space by $u_1 = P(x,t)$, $u_2 = u_2(x,t), \ldots,$ $u_n = u_n(x,t)$, $t = t$. This is permissible, since $\frac{\partial(u,t)}{\partial(x,t)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \vdots \\ 0 \cdots 0 & 1 \end{vmatrix} = \frac{\partial u}{\partial x} \neq 0$. Now let $\varphi(x)$ be a test function with support in the domain where the u-coordinates are valid, and set $A(t) = (\delta^{(k)}(P_t), \varphi) =$ $(-1)^k \int \partial^k(\varphi \frac{\partial x}{\partial u})/\partial u_1^k \Big|_{u_1=0} du_2 \ldots du_n$ where the right-hand side is calculated separately for each t. The question is, what is $A'(t)$. To calculate this, consider $A(t)$ as a distribution on $a < t < b$. If $\psi(t)$ is a C^{∞} function vanishing outside $a < t < b$, then

$$(A'(t),\psi) = -\int_a^b A(t)\,\psi'(t)dt =$$

$$(-1)^{k+1}\int_a^b \psi'(t)\int \partial^k(\varphi\left|\frac{x}{u}\right|)/\partial u_1^k\Big|_{u_1=0} du_2\ldots du_n dt$$

$$=(-1)^{k+1}\int_a^b \partial^k(\frac{\partial\varphi\psi}{\partial t}\left|\frac{\partial(x,t)}{\partial(u,t)}\right|)/\partial u_1^k\Big|_{u_1=0} du_2\ldots du_n dt$$

$$= -(\delta^{(k)}(P(x,t)),\ \partial\varphi\psi/\partial t) = (\delta^{(k+1)}(P(x,t))\ \partial P/\partial t, \varphi\psi)$$

$$=(-1)^{k+1}\int_a^b \partial^{k+1}(\varphi\psi\frac{\partial P}{\partial t}\left|\frac{\partial(x,t)}{\partial(u,t)}\right|)/\partial u_1^k\Big|_{u_1=0} du_2\ldots du_n dt$$

$$= \int_a^b (\delta^{(k+1)}(P_t)\ \partial P/\partial t, \varphi)\ \psi(t)dt.$$

Thus

$$\underline{\partial\delta^{(k)}(P_t)/\partial t = \delta^{(k+1)}(P_t)\ \partial P_t/\partial t,}$$

the same formula as before, but with a slightly different meaning.

§ 8. Solution of Cauchy's problem for the wave equation in odd spatial dimensions.

Let $r^2 = \sum_1^n x_j^2$, and consider t a parameter.

By virtue of (§ 5) and (§ 7), we have

$$\partial^2\delta^{(k)}(r^2-t^2)/\partial x_j^2 = 2\delta^{(k+1)}(r^2-t^2)+4x_j^2\ \delta^{(k+2)}(r^2-t^2),\ \text{and}$$

$$\partial^2\delta^{(k)}(r^2-t^2)/\partial t^2 = -2\delta^{(k+1)}(r^2-t^2)+4t^2\ \delta^{(k+2)}(r^2-t^2).$$

Thus

$$\sum_1^n \partial^2\ \delta^{(k)}(r^2-t^2)/\partial x_j^2 - \partial^2\delta^{(k)}(r^2-t^2)/\partial t^2 =$$
$$2(n+1)\delta^{(k+1)}(r^2-t^2)$$

$+4(r^2-t^2)\delta^{(k+2)}(r^2-t^2)$. By (§ 6), this can be rewritten

$\Box\, \delta^{(k)}(r^2-t^2)=2\,[n-2k-3]\,\delta^{(k+1)}(r^2-t^2)$, with $\Box$ the wave operator. Hence if $k=\frac{n+3}{2}$, $\Box\,\delta^{(k)}(r^2-t^2)=0$ for $t>0$ and for $t<0$.. We investigate next its initial values $\lim_{t\to 0} \delta^{(k)}(r^2-t^2)$ and $\lim_{t\to 0} \partial\delta^{(k)}(r^2-t^2)/\partial t$.

We have from (4,iii)

$$(\delta^{(k)}(r^2-t^2),\varphi)=(-1)^k\,\frac{1}{2}\int_\Omega \left(\frac{1}{2r}\frac{\partial}{\partial r}\right)^k(\varphi r^{n-2})\Big|_{r=t}\,d\Omega\,.$$

Since $k=\frac{n-3}{2}$, k operations by $\frac{1}{2r}\frac{\partial}{\partial r}$ reduce the power of r in φr^{n-2} by $2k=n-3$. Thus

$\left(\frac{1}{2r}\frac{\partial}{\partial r}\right)^k(\varphi r^{n-2})\Big|_{r=t} = O(t)$, and $\lim_{t\to 0}(\delta^{(k)}(r^2-t^2),\varphi)=0$.

For the first derivative, expand

$\varphi(r\omega)=\varphi(0)+r\,\varphi_1(\omega)+r^2\,\varphi_2(\omega)+\ \ldots$, which leads to

$$\left(\frac{1}{2r}\frac{\partial}{\partial r}\right)^{k+1}(r^{n-2}\,\varphi(r\omega))=2^{-k-1}\varphi(0)\,\frac{(n-2)\ldots(n-2k-2)}{r}$$
$$+\ c_1\,\varphi_1(\omega)\ +\ c_2 r\,\varphi_2(\omega)+\ \ldots\ .$$

Thus

$$\lim_{t\to 0}\frac{\partial}{\partial t}(\delta^{(k)}(r^2-t^2),\varphi)=\lim_{t\to 0}(-2t)\ (\delta^{(k+1)}(r^2-t^2),\varphi)$$
$$=\lim\ (-1)^k t\int_\Omega \left(\frac{1}{2r}\frac{\partial}{\partial r}\right)^{k+1}(\varphi r^{n-2})\Big|_{r=t}\,d\Omega =$$
$$(-1)^k|\Omega|\,\frac{(n-2)!}{(\frac{n-3}{2})!}\,\varphi(0)$$
$$=(-1)^k 2\pi^{k+1}(\delta(x),\varphi),\ \text{or}$$

$$\lim_{t\to 0}\ \partial\delta^{(k)}(r^2-t^2)/\partial t\ =\ (-1)^k\ 2\pi^{k+1}\ \delta(x).$$

This solves the Cauchy problem for a unit pulse in the time derivative at $t=x=0$. To solve $\Delta v-v_{tt}=0$ with more normal initial values, e.g. $v(x,0)=0$, $v_t(x,0)=\varphi$, we can set

$$v(x_o,t)=\frac{(-1)^k}{2}\,\pi^{-k-1}\ (\delta^{(k)}(r^2-t^2),\ \varphi(x_o+x))=$$
$$\frac{(-1)^k}{2}\,\pi^{-k-1}\ \delta^{(k)}(r^2-t^2)*\varphi\ .$$

§ 8'. Extension of the solution to even spatial dimensions. The previous solution leads easily to the case of even dimensions by Hadamard's method of descent. To solve the equation $\Box v=0$, $v=0$ if $t=0$, $v_t=\varphi(x_1,\dots,x_n)$ if $t=0$, with n even, define first a function $\tilde{\varphi}(x_1,\dots,x_{n+1})=\varphi(x_1,\dots,x_n)$, let $k=\frac{n+4}{2}$, and set $v(x_1,\dots,x_{n+1})=\frac{(-1)^k}{2}\pi^{-k-1}\gamma^{(k)}(\rho^2-t^2)*\tilde{\varphi}$, with $\rho^2=\sum_1^{n+1}x_j^2$. Then $\partial v/\partial x_{n+1}=\delta^{(k)}(\rho^2-t^2)*\partial\tilde{\varphi}/\partial x_{n+1}=0$, so $\sum_1^n\partial^2v/\partial x_j^2-\partial^2v/\partial t^2=\sum_1^{n+1}\partial^2v/\partial x_j^2-\partial^2v/\partial t^2=0$. Also $\lim_{t\to 0}v=0$, and $\lim_{t\to 0}\partial v/\partial t=\varphi$. Thus v solves the wave equation in n dimensions.

For an explicit form of the solution, let $\omega=(\omega_1,\dots,\omega_{n+1})$ be on the unit sphere Ω_{n+1} in R^{n+1}, and $\omega'=(\omega_1,\dots,\omega_n,0)$. Then

$$v(x_1,\dots,x_n,t)=(-1)^k\,\tfrac{1}{2}\int_{\Omega_{n+1}}\left(\frac{1}{2\rho}\frac{\partial}{\partial\rho}\right)^k(\varphi(x+\omega'\rho)\rho^{n-1})\Big|_{\rho=t}\,d\Omega_{n+1}.$$

If $\xi=(\xi_1,\dots,\xi_n)$ is a variable point in R^n, this becomes $v=(-1)^k\int_{|\xi|\le 1}\left(\frac{1}{2\rho}\frac{\partial}{\partial\rho}\right)^k(\varphi(x+\rho\xi)\rho^{n-1})\Big|_{\rho=t}\frac{d\xi}{\sqrt{1-|\xi|^2}}$. Since v depends on all values of φ inside (not just on) the light cone, the "strong Huygens' principle" is lacking.

§ 9. Further formulas

i) For distributions f we have, for a change of coordinites u(x), that $\partial f/\partial u_i = \sum \partial f/\partial x_j \ \ \partial x_j/\partial u_i$. Thus
$\partial \delta^{(k)}(P)/\partial u_j = \delta^{(k+1)}(P)\ \partial P/\partial u_j$, $k = -1, 0, 1, \ldots$.

For example in spherical coordinates with $r > 0$ we have, for $t \neq 0$, $\theta(r^2-t^2) = \theta(r-t) - \theta(-r-t)$, so that $2r\delta(r^2-t^2) = \delta(r-t) + \delta(-r-t) = \delta(r-t) + \delta(r+t)$, or $\delta(r^2-t^2) = (2r)^{-1}\,[\delta(r-t) + \delta(r+t)]$. Clearly one of $\delta(r-t)$ and $\delta(r+t)$ is zero, according as $t < 0$ or $t > 0$.

ii) If P and Q have no common zeroes,

$$(\delta(PQ), \varphi) = \int_{PQ=0} \frac{\varphi}{|\nabla PQ|}\, d\sigma = \int_{P=0} \frac{\varphi}{|Q||\nabla P|}\, d\sigma + \int_{Q=0} \frac{\varphi}{|P||\nabla Q|}\, d\sigma =$$

$\left(\frac{\delta(P)}{|Q|} + \frac{\delta(Q)}{|P|}, \varphi\right)$. If Q has no zeros at all, and $Q > 0$, then $\delta(PQ) = Q^{-1}\delta(P)$. We can apply this to $r^2-t^2 = (r-t)(r+t)$ for $t > 0$, finding $\delta(r^2-t^2) = \frac{\delta(r-t)}{r+t} = \frac{\delta(r-t)}{2r}$.

iii) The above formula can be taken further by differentiation, but the results are not neat. If $Q > 0$ everywhere, however, it works nicely. Differentiating $Q\delta(PQ) = \delta(P)$, we have

$$Q_j\delta(PQ) + Q\delta'(PQ)PQ_j + Q^2\delta'(PQ)P_j = \delta'(P)P_j \ .$$

By (§ 6), the first two terms cancel, and the same process as in (§ 6) justifies cancelling P_j from the resulting equation, so

$$Q^2\ \delta'(PQ) = \delta'(P) \ .$$

Proceding by induction,

$$\underline{Q^{k+1}\ \ \delta^{(k)}(PQ) = \delta^{(k)}(P)},$$

if $Q > 0$ everywhere.

Thus e.g. $\delta^{(k)}(r^2-c^2) = (r+c)^{-k-1}\ \delta^{(k)}(r-c)$, for $c > 0$.

§ 10. Other generalizations of $\delta(x)$.

There is another sort of δ-function that generalizes the 1-dimensional $\delta(x)$, and that provides a convenient notation frequently used in applied mathematics. Originally we interpreted $(\delta(x), \varphi(x)) = \varphi(0)$ as the (0-dimensional) integral of $\varphi(x)$ over the set x=0, which led to the interpretation of $(\delta(x_1), \varphi(x_1,x_2))$ as the 1-dimensional integral $\int_{-\infty}^{\infty} \varphi(0,x_2)dx_2$ over the set $x_1=0$, and thence to the $\delta(P)$ of § 2. However $\delta(x)$ could just as well be viewed as a restriction map, transforming functions on the line to functions on the point x=0. From this point of view, the generalization to two dimensions would require $\delta(x_1)\cdot\varphi(x_1,x_2) = \varphi(0,x_2)$. To avoid confusion, we denote this operation by a different symbol, $\delta_o(x_1)$. Thus $\delta(x_1)$ is a distribution, but $\delta_o(x_1)$ is a mapping from test functions of n variables to test functions of n-1 variables, defined by $\delta_o(x_1)\ \varphi(x_1,\ldots,x_n) = \varphi(0,x_2,\ldots,x_n)$. Such a mapping might be called a <u>partial distribution</u>. If the test functions are topologized in any of the standard ways, e.g. as $K(M_p)$, then $\delta_o(x_1)$ is clearly continuous, and of course linear.

The adjoint of $\delta_o(x_1)$, which we denote by $\delta^o(x_1)$, is then a continuous map from distributions on R_{n-1} to distributions on R_n. If g is a distribution on R_{n-1}, then $\delta^o(x_1)g$ is defined by $(\delta^o(x_1)g, \varphi) = (g, \delta_o(x_1)\varphi)$. In terms of this new concept we can write in a well-defined way $\delta(x_1,\ldots,x_n) = \delta^o(x_1)\ldots\delta^o(x_{n-1})\delta(x_n)$; the usual expression written, i.e. $\delta(x_1,\ldots,x_n) = \delta(x_1)\ldots\delta(x_n)$ does not allow one to interpret $\delta(x_j)$ in the same sense as the general $\delta(P)$ defined in § 2.
The connection between $\delta^o(x_1)$ and $\delta(x_1)$ can be expressed by $\delta(x_1) = \delta^o(x_1)\ \{1\}$, where $\{1\}$ is the distribution on R_{n-1} given by $(\{1\}, \varphi) = \int \varphi(x_2,\ldots,x_n)dx_2\ldots dx_n$.

Similarly $\delta^k(x_1)\varphi$ is the restriction to $x_1=0$ of $(-\frac{\partial}{\partial x_1})^k\varphi$. Thus $\delta^k(x_1)\ \{1\} = \delta^{(k)}(x_1)$. It is also easy to see that $x_1\,\delta^k(x_1) = -k\,\delta^{k-1}(x_1)$. However it is apparently impossible to make any analogy with the formula $\partial\delta^{(k)}(P)/\partial x_j = \partial P/\partial x_j\ \delta^{(k+1)}(P)$; there seems to be no reasonable way of defining $\partial(\delta^k(x_1))/\partial x_2$ so that $\partial\delta^k(x_1)/\partial x_2 = 0 = \delta^{k+1}(x_1)\ \partial x_1/\partial x_2$. The trouble is that the formula for $\delta^{(k)}$ depends on an integration by parts, and in δ^k there is no integration.

In § 11 we define a $\delta_k(P)$ that generalizes $\delta_k(x_1)$, and satisfies $P\ \delta_k(P) = -k\ \delta_{k-1}(P)$. It follows from this that for any distributions $g_o,\dots,g_N$ on $\{P=0\}$, $f=\sum_0^N \delta^k(P)g_k$ satisfies $P^{N+1}f=0$. Proposition 1 of § 11 shows that conversely every solution of $P^{N+1}f=0$ has the form $f=\sum_0^N \delta^k(P)g_k$. Thus any distribution f such that $x_1^{N+1}f=0$ can be obtained by applying to f and its first N normal derivatives N+1 distributions in the plane $x_1=0$.

§ 11 $\delta^k(P)$.

Let P be a C^∞ function such that $|P|^2+|\nabla P|^2 > 0$. Then the set $\{P=0\}$ is a C^∞ Riemannian manifold, and distributions on $\{P=0\}$ can be defined as continuous functionals on K(P=0), the space of all C^∞ functions of compact support defined on $\{P=0\}$. Each such test function is the restriction to $\{P=0\}$ of a test function in $K(R_n)$. The topology of K(P=0) can be given in terms of a particular way of extending functions on $\{P=0\}$ to functions on R_n (which we shall indicate in § 13), and requiring this to be a homeomorphism. Another way is to define a sequence of norms in K(P=0) by $\|\phi\|_k = \inf \|\psi\|_k$, where ψ is a C^∞ function ofcompact support on R_n with $\psi=\varphi$ on $\{P=0\}$, and the inf is taken over all such ψ.

The "partial distribution" $\delta_k(P)$ is then a continuous map from $K(R_n)$ to K(P=0) obtained as follows. For φ in $K(R_n)$ let $L\varphi = \frac{\nabla P.\nabla\varphi}{|\nabla P|^2}$, defined in a neighbourhood of $\{P=0\}$; in a sense to be made precise in § 12, $L\phi = -\partial\phi/\partial P$. This suggests an identity that can easily be proved from the actual definition of L,

1) $L^k P\varphi = -k\, L^{k-1}\varphi + PL^k\varphi$.

Then we make the definition

2) $\delta_k(P)\varphi = L^k\varphi$ restricted to $\{P=0\}$.

From (1) follows immediately

3) $\delta_k(P)P\varphi = -k\,\delta_{k-1}(P)\varphi$,

which corresponds to the formula of § 6. As we have seen, the more precise formula of § 5 cannot be expected to apply to $\delta_k(P)$.

If g is a distribution on $\{P=0\}$, then $\delta^k(P)g$ is the distribution on R_n defined by

4) $(\delta^k(P)g,\varphi)=(g,\delta_k(P)\varphi)$.

Thus $\delta^k(P)$ is the adjoint of $\delta_k(P)$.

From (3) it follows that for an arbitrary distribution g on $\{P=0\}$

5) $P\ \delta^k(P)g = -k\,\delta^{k-1}(P)g$.

Thus $P^N\delta^k(P)g = 0$ for $k < N$: $(P^N\delta^k(P)g,\varphi)=$

$(g,\delta_k(P)\,P^N\varphi)=(g,(-1)^k\,k!\ \delta_0(P)\,P^{N-k}\varphi)=(g,0)=0.$

Conversely, the following result holds.

<u>Proposition 1</u>. <u>If f is a distribution on R_n such that $P^{N+1}f=0$, then there are unique distributions</u> $g_0,\dots,g_N$ <u>on $\{P=0\}$ such that</u> $f=\sum_{k=0}^{N}\delta^k(P)g_k$.

The proof will be indicated in § 13.

<u>Examples</u>.

i) $P=r-c$, $|\nabla P|=1$, $L\varphi = -\partial\varphi/\partial r$,

$\delta_k(r-c)\varphi = (-\frac{\partial}{\partial r})^k\varphi\Big]^{r=c}$.

ii) $P=r^2-c^2$, $|\nabla P|= 2r$, $L\varphi = -\frac{1}{2r}\frac{\partial\varphi}{\partial r}$,

$\delta_k(r^2-c^2)\varphi = (-\frac{1}{2r}\frac{\partial}{\partial r})^k\varphi\Big]^{r=c}$.

iii) $P=x_1^2-\sum_2^n x_j^2 -m^2=\sum g_{jj} x_j^2-m^2$, $(\nabla P)_j=2g_{jj}x_j$,

$$\frac{\nabla P.\nabla\varphi}{|\nabla P|^2} = \tfrac{1}{2}|x|^{-2}\sum g_{jj}x_j\,\partial\varphi/\partial x_j,$$

$$\delta_0(P)\varphi=\varphi\Big]^{P=0},\quad \delta_1(P)\varphi=-\tfrac{1}{2}|x|^{-2}\sum g_{jj}x_j\,\partial\varphi/\partial x_j\Big]^{P=0},$$

but formulas for $\delta_k,\ldots$ become messy. According to Proposition 1, every solution of

6) $(\sum g_{jj}x_j^2-m^2)f=0$

is of the form $f=\delta^0(P)g$, with g a distribution on $\{P=0\}$. Equation (6) is the Fourier transform of the Klein-Gordon equation $[-(\partial/\partial y_1)^2+\sum_2^n(\partial/\partial y_j)^2-m^2]\tilde{f}=0$, important in quantum field theory. Some properties of g can be deduced from corresponding properties of $\tilde{f}$; e.g. if $\tilde{f}$ leads to a continuous energy tensor, then g is a locally square integrable function on $\{P=0\}$.

Remark. The $\delta_k(P)$ defined above is clearly not the only generalization of $\delta_k(x_1)$; for example the restriction of $L^k(\varphi/|\nabla\varphi|)$ to $\{P=0\}$ would be another candidate, and would lead to an obvious connection between $\delta_0(P)$ and $\delta(P)$, i.e. $(\delta(P),\varphi)=\int_{P=0}\delta_0(P)\varphi\,d\sigma$. Definition (2) has been chosen as the simplest expression for which (3) holds. In § 12 we show that a partial distribution $\delta^k(P)$ can be defined so that for each k $(\delta^{(k)}(P),\varphi)=\int_{P=0}\delta^k(P)\varphi\,d\sigma$; but it is practically impossible to calculate $\delta^1(P)$ explicitly for the P of example (iii) above.

§ 12. Canonical coordinates

Here we introduce in a neighbourhood U of $\{P=0\}$ new coordinates (ξ,t) (ξ in $\{P=0\}$, t real) such that t=P and the curves ξ = constant are orthogonal to all the surfaces P=constant. The existence of such a system rests on ordinary differential equations, as follows.

Through each point x_0 in $\{|\nabla P| > 0\}$ there is a unique solution $a(x_0,t)$ of $da/dt = \nabla P/|\nabla P|^2$ with the initial values $a(x_0,P(x_0))=x_0$. There is a neighbourhood $U(x_0)$ and a number $\varepsilon(x_0)$ such that the solution $a(x,t)$ is defined for all x in $U(x_0)$ and all $|t-P(x_0)| < \varepsilon$, and is a C^∞ function (x,t) there. Since $P(a(x,P(x)))=P(x)$ and $dP(a(x,t))/dt = \nabla P.da/dt=1$, it follows that $P(a(x,t))=t$.
Thus for fixed x the curve $a(x,t)$ is orthogonal to P=constant, and the parameter t is P. Now there is an open set U containing $\{P=0\}$ such that for each x in U $a(x,t)$ is defined for $|t| \leq 2\,|P(x)|$. In U is defined the function $a(x,0)$, which is thus a C^∞ map of U onto $\{P=0\}$.

We call the pair $(a(x,0),P(x))$ <u>canonical coordinates</u> of the point x in U; the coordinates are actually a map onto an open subset V of the Cartesian product of $\{P=0\}$ by the real line. If ξ represents an arbitrary point on $\{P=0\}$ and t is a real number, then for all (ξ,t) in V the map $(\xi,t) \to a(\xi,t)$ is the inverse of $x \to (a(x,0),P(x))$. Since P=t in this correspondence, it makes sense for C^∞ functions φ defined in U to let $\partial\varphi/\partial P = d\varphi(a(\xi,t))/dt = \nabla\varphi.da/dt$ $= \nabla\varphi.\nabla P/|\nabla P|^2$. This is the interpretation of $L\varphi$ mentioned in § 11.

We can further define a partial distribution $\delta_k(P)$ (probably of no practical importance) such that $\int_{P=0} \delta_k(P)\varphi \; d\sigma = (\delta^{(k)}(P),\varphi)$. To this end let v(y) be the function such that for each continuous φ with compact support, $\int_{P=t} \varphi(y)d\sigma_t = \int_{P=0} \varphi(a(y,t))v(a(y,t))d\sigma$, where $d\sigma_t$ is the surface element on $\{P=t\}$ and $d\sigma = d\sigma_0$. It is easy to check that such a v exists, is C^∞, and is unique. Clearly $v(y)=1$ if y is on $\{P=0\}$.

We now define

$$7)\quad \delta_k(P)\varphi = \left(-\frac{\partial}{\partial P}\right)^k (\varphi v/|\nabla P|)\Big]^{P=0} .$$

It is trivial to prove by induction that

$$(\delta^{(k)}(P+s),\varphi)=\int_{P=0}(d/dt)^k\left(\frac{\varphi v}{|\nabla P|}(a(\xi,-t))\right)\Big]^{t=s}d\sigma\ ,$$

and consequently that

$$8)\quad (\delta^{(k)}(P),\varphi)=\int_{P=0}\underline{\delta}_k(P)\varphi\ \ d\sigma\ .$$

This $\underline{\delta}_k(P)$ is relatively easy to calculate if $v(y)$ is easy to calculate (e.g. if the surfaces P=constant are spheres, cylinders, or planes), but otherwise quite difficult.

§ 13. Characterization of the solutions of $P^N f=0$

Before proving Proposition 1, we give a way of extending functions in $K(P=0)$ to functions in $K(R_n)$, in terms of the canonical coordinates of § 12. Define a C^∞ "cut-off function" $\chi(x)$ such that $\chi\equiv 1$ in a neighbourhood of $\{P=0\}$, $\chi(x)=0$ if the distance from x to $\{P=0\}$ is $\geqq 1$, and the support of χ is contained in the neighbourhood U where the canonical coordinates are defined. Such a χ can be obtained by defining it successively in the spheres $|x|\leqq n$. Then if ψ is in $K(P=0)$ $\tilde{\psi}(x)=\chi(x)\psi(a(x,0))$ is in $K(R_n)$; and on $\{P=0\}$, $\psi=\tilde{\psi}$. Here $a(x,t)$ is the function of § 12.

Proposition 1 is proved by induction. If f is a distribution on R_n such that $Pf=0$, then a distribution g_o on $\{P=0\}$ must be found such that $f=\delta^o(P)g$. If ψ is in $K(P=0)$, let $\tilde{\psi}$ be the extension indicated above, and define $(g_o,\psi)=(f,\tilde{\psi})$. To show that $f=\delta^o(P)g_o$, note that $(f,\varphi)=(f,\chi\varphi)$, so that $(f,\varphi)-(\delta^o(P)g_o,\varphi)=(f,\chi(x)\ \varphi(x))-$ $(f,\chi(x)\ \varphi(a(x,0))=(Pf,\chi(x)\ \dfrac{\varphi(x)-\varphi(a(x,0))}{P(x)}\ )$. Since $Pf=0$, this expression is zero if

$\chi(x)\ \dfrac{\varphi(x)-\varphi(a(x,0))}{P(x)}$ is a test function. The fact that it has compact support follows from the properties of χ, so the only question is its differentiability. In canonical coordinates (ξ,t) let $\varphi(x)=\varphi_1(\xi,t)$, where $(\xi,t)=(a(x,0),P(x))$; and $\varphi_2(\xi,t)=d\ \varphi_1(\xi,t)/dt$. Then φ_1

and φ_2 are in C^{∞} on the support of χ , and

$$\frac{\varphi(x)-\varphi(a(x,0))}{P(x)} = \frac{\varphi_1(\xi,t)-\varphi_1(\xi,0)}{t} = \frac{1}{t}\int_0^1 \frac{d}{ds}\varphi_1(\xi,ts)ds$$

$= \int_0^1 \varphi_2(\xi,ts)ds$, which is in C^{∞} on the support of χ .

This establishes Proposition 1 for N=0. Suppose for all $N < M$ the solution of $P^{N+1}f=0$ can be represented as $\sum_0^N \delta^k(P)g_k$, and let now $P^{M+1}f=0$. Then there is a g_M such that $P^M f=(-1)^M M!\,\delta^o(P)g_M$. In view of formula (5) of § 11, $P^M(f-\delta^M(P)g_M)=P^M f-(-1)^M M!\,\delta^o(P)g_M=0$, so that by the induction assumption there are $g_1,\dots,g_{M-1}$ such that $f-\delta^M(P)g_M=\sum_0^{M-1}\delta^k(P)g_k$.

The uniqueness is easy to check for M=0, and is then extended to other M by formula (5).